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ABSTRACT

This curriculum guide covers Course II of a three-year sequence for high school mathematics in New York which was intended to provide an alternative to the regular Regents' sequence of ninth-, tenth-, and eleventh-grade mathematics. A listing of scope and content is provided along with suggested time allotments. Topics covered are: Logic, Mathematical Systems, Euclidean Geometry, Analytic Geometry, Solutions of Quadratic Equations, Probability, Permutations, Combinations, and possible optional topics. Material in each chapter is discussed and some teaching suggestions are made. (MP)

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THREE-YEAR SEQUENCE FOR HIGH SCHOOL

mathematics

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COURSE II

SE 032 779

The University of the State of New York
THE STATE EDUCATION DEPARTMENT
Bureau of General Education Curriculum Development
Albany, New York

SEP 2 2 1980

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FOREWORD

This is the second in a series of three publications designed for the new *Three-Year Sequence for High School Mathematics* program consisting of three unified courses - *Course I, Course II, and Course III*. Currently, this sequence provides an alternate to the regular Regents sequence of ninth, tenth, and eleventh year mathematics. Examinations will be available each year for each course, and students who successfully complete the courses will be credited with a Regents sequence in mathematics.

In June 1972, an Advisory Committee was convened by Frank Hawthorne, then Chief of the Bureau of Mathematics Education, to consider the development of a three-year high school mathematics program which would bring together the various branches of mathematics that historically have been treated independently. Members of this committee included: Robert Atkins, John Adams High School, New York City; Donald Holquist, then of Mineola High School, now at Baldwin High School; Anthony Prindle, Linton High School, Schenectady; Ralph Roberts, Pelham Memorial High School; Harry Ruderman, Hunter College High School; George Salterelli, Amherst High School; Stephen Willoughby, New York University; and Sister Ann Xavier, Nazareth High School, Rochester.

Using the philosophical deliberations and tentative outline developed by the committee, *Course II* was organized and written by Sister Ann Xavier of the committee and Richard Klutch of Hunter College High School, working under the direction of Fredric Paul, Chief of the Bureau of Mathematics Education. Elliot Lamb of the Levittown schools reviewed the material. In September 1975, 40 school districts began offering the course.

As a result of a series of meetings with classroom teachers during 2 years of teaching the course, together with their written evaluations at the end of that time, *Course II* was revised during the summer of 1977. Maureen Barry of Trumansburg High School, Ellen Lewis of Mount Vernon High School, and Sherman Taishoff of Harrison High School, analyzed the evaluations and made recommendations which were incorporated into the revision of the materials produced by the original writers.

The final draft was prepared and edited by Mr. Paul and Aaron Buchman, Associate in Mathematics Education. Robert F. Zimmerman, Associate in Secondary Curriculum, represented the Bureau of General Education Curriculum Development on the committee and served as the liaison with the Bureau of Mathematics Education. The final manuscript was prepared in the Bureau of General Education Curriculum Development.

October 1977

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COURSE II

Scope and Content

I. Logic (recommended time: 20 - 25 days)

A. Review of material on logic from Course I

B. Laws of reasoning: Laws of Contraposition, Detachment, Syllogism, and Disjunctive Inference; negation of conjunction, disjunction (De Morgan's Laws); applications

C. Quantification: universal and existential quantifiers; notation; negation of quantified statements; counterexample

II. Mathematical Systems (recommended time: 15 - 20 days)

A. Investigation of finite mathematical systems: computation and equation solving; binary operation; identity and inverse; closure; commutativity; and associativity in these systems

B. Groups: definition, reinvestigation, and reclassification of systems studied thus far

C. Discussion of group theorems and applications

1. cancellation law
2. $a \circ x = b \rightarrow x = a^{-1} \circ b$

D. Fields

1. Review of the distributive property
2. Definition, examples, and applications of fields

III. Euclidean Geometry (recommended time: 55 - 60 days)

A. Discussion of the foundations of an axiomatic system: undefined terms, definitions, axioms, and theorems.

B. Discussion of specific terms and assumptions in proofs of theorems

1. Betweenness: definition or acceptance as an undefined term
Congruence: definition and assumptions (ASA, SAS, SSS)
Theorem: *The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle.*

Theorem: *The base angles of an isosceles triangle are congruent.*

2. Parallel lines: definition and assumptions
Theorem: *If two parallel lines are cut by a transversal, then the alternate interior angles are congruent and conversely.*

Theorem: *The sum of the measures of the angles of a triangle is 180.*

Theorem: *Two triangles are congruent if two angles and the side opposite one of them are congruent to the corresponding angles and side of the other.*

Theorem: *If two angles of a triangle are congruent, then the sides opposite these angles are congruent (converse of second theorem listed).*

3. Similarity: definition and assumptions
Assumption or Theorem: *If a line is parallel to one side of a triangle and intersects the other two sides, it divides the other two sides into segments whose lengths are proportional, and conversely.*

Theorem: *Two triangles are similar if two angles of one are congruent to two angles of the other.*

Theorem: *If an altitude is drawn to the hypotenuse of a right triangle: (a) the two triangles formed are each similar to the original and to each other; (b) the length of each leg is the mean proportional between the length of the hypotenuse and the length of the projection on the hypotenuse; and (c) the length of the altitude is the mean proportional between the length of the segments of the hypotenuse.*

Theorem: *If a triangle is a right triangle, then the square of the length of the longest side is equal to the sum of the squares of the lengths of the other two sides, and conversely. (Pythagorean theorem)*

C. Constructions

1. construct a segment congruent to a given segment
2. bisect a segment
3. construct an angle congruent to a given angle
4. bisect an angle
5. construct a perpendicular to a given line through a point on or outside the line
6. construct an altitude and a median to a given side of a triangle
7. construct a line parallel to a given line through a point not on the line
8. construct a triangle similar to a given triangle on a given base

9. divide a segment into k congruent parts (optional)
10. construct a circle circumscribed about a given triangle

D. Equivalence relations (optional)

1. reflexive, symmetric, and transitive properties
2. parallelism, similarity, congruence, equality, etc., as examples of equivalence relations

IV. Analytic Geometry (recommended time: 25 - 30 days)

A. Review of material on rectagular coordinate system and graphing from Course I

B. Distance formula, midpoint, of a line segment, division of a line segment, and problems involving area

C. Slopes of parallel and perpendicular lines

D. Equations of a straight line: point-slope and slope-intercept forms

E. Locus: definition; locus of points equidistant from two points, three points, two parallel lines, at a fixed distance from a point (circle), equidistant from a fixed point and a fixed line (parabola); equations of circle and parabola.

F. Proof by analytic means

V. Solution of Quadratic Equations (recommended time: 10 - 15 days)

A. Review of the solution of quadratic equations by factoring; solution by graphing

B. Completing the square

C. Derivation of the quadratic formula

D. Graphical and algebraic solution of a quadratic and linear equation simultaneously

E. Application to cases with rational and irrational roots

VI. Probability, Permutations, and Combinations (recommended time: 10 - 15 days)

A. Review of material on permutations from Course I

B. Permutations of n things taken n at a time with repetition

C. Combinations of n things taken n at a time, r at a time

Notation: ${}_nC_r$, $C(n,r)$, $\binom{n}{r}$; that ${}_nC_r = {}_nC_{n-r}$

D. Probablity involving combinations

VII. Possible Optional Topics

A. Binomial theorem

B. Space visualization

C. Number theory

D. Introduction to trigonometry (especially for students terminating with Course II)

E. Matrices

F. Computers

INTRODUCTION

This booklet presents an outline and notes for the second of three courses of a new program of mathematics for the high school student. One of the goals of the program is to integrate the various branches of mathematics (e.g., algebra, geometry, trigonometry) heretofore treated as independent year- or semester-long courses by most schools in the State. In addition, there are some "new" topics; these have been combined with important "traditional" topics in a manner which emphasizes their interdependence. Of course, the so-called "new" topics are new only in their placement in the high school curriculum, not in the whole scheme of mathematics.

Trial runs indicate that the material presented here can be handled comfortably by most classes in a normal school year, and it is assumed that this year will be the tenth. (The estimated time for the entire program is three years: 9, 10, 11.) Although this course is written with tenth grade students in mind, it is possible that some students may begin the material in an earlier or later year, or complete it in less or more than one year. It is expected that individual teachers will consider their students' abilities and interests in making judgments concerning time allotments, aspects of the material to be stressed or extra material to be presented. To aid in some of these decisions, an attempt has been made to indicate many areas of possible de-emphasis on the one hand and of enrichment on the other. Some of these optional parts are extensions of the material being taught and are included in the course notes. Further optional topics are suggested in the last section of this outline, although individual teachers may find other topics more suitable. It should be noted, as was the case in Course I, that topics need not be taught in the order given in this outline, that extra topics need not be limited to those suggested here, and that no particular teaching approach to any topic is required.

For teachers who think of tenth grade mathematics as a geometry course, this course may seem a radical change. But it is not so radical. An attempt has been made to place the learning of important geometric concepts in all three years of the program, and, in so doing, relate them more usefully to other concepts. In Course I, students met many important geometric concepts (congruence and similarity, for example) on an informal basis; they are now ready to begin Course II with a comfortable familiarity with these topics. The geometry of Course II, then, built around a short, deliberately skeletal, suggested set of theorems, is intended both to extend the knowledge of these geometric relationships and to present a model of the application of logical reasoning to the development of a set of theorems from basic axioms and postulates. It is hoped that the shorter sequence of theorems will help to emphasize their logical development. These theorems center about congruence and similarity of triangles and the basic theorems about quadrilaterals. A fuller discussion about the choice of theorems appears in the notes.

In Course I students learned to solve quadratic equations with rational roots by factoring. However, a false impression is created if none of the problems a student encounters involving quadratic equations has irrational solutions. It seems, then, that the quadratic formula should be derived and used in Course II. Further, the graph of $y = ax^2 + bx + c$ is a reasonable extension of the graphing in Course I as well as the work with locus in Course II.

So far, all the students' work with the real number field has been informal. The section here on mathematical systems is intended to introduce, first in a finite situation, the basic group and field properties which may be applied more formally to the real and complex numbers in Course III. The section on systems can be taught as abstractly as the ability and interests of students will justify.

The work with logic and with probability and combinatorics introduced in Course I is reviewed and extended. Needless to say, it is hoped that the work with logic will permeate all parts of this course as well as Course III, where no further formal section of logic is included. The teacher is urged to foster constant awareness of the interdependence of all the topics considered.

As in Course I, there are no formal units on the computer or the calculator in this outline. Nonetheless, the use of calculators, computers, and flow-charting is suggested in those areas where such devices can contribute to mathematical understanding, increase student participation and enjoyment, or broaden the area from which applications of mathematical ideas can be selected. Individual teachers may wish to give lesser or greater emphasis to the role which these modern components can play in mathematics study.

It is believed that the program outlined is honest and coherent, and that it reflects what is happening in mathematics today. It is important to reaffirm that one of the goals of the program is to give students a useful and enjoyable experience with mathematics which may encourage them to continue their mathematical studies. It is hoped that students may experience the excitement and beauty of mathematics as fully as their abilities permit.

I. Logic

A. After a review of the logical connectives learned in grade 9 ($\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$) and a review of the use and significance of truth tables, students should be prepared to begin a study of logical *reasoning*. As an example, consider the process of determining, in a logical manner, if certain statements are necessarily true or false, given the truth or falsity of certain other statements. The processes by which these conclusions are reached can be conveniently summarized as *rules of logic*, or *rules of inference*. (That some of the names of these rules are still in the original Latin indicates that the systematic study of logic is not as "new" as some people might claim.) Some of these methods of reasoning occur so frequently, and are so useful, that they merit study at this time. The order in which these rules are presented is not significant, and will depend on teacher preference, or, perhaps, the outcome of a class discussion.

B. The *Law of the Contrapositive*, or of *Contraposition*, is simply a formalization of previous work with the contrapositive of a conditional. It states that a conditional and its contrapositive always have the same truth value; i.e., if a conditional is true, then its contrapositive is also true and if a conditional is false then its contrapositive is also false. In symbols, $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$, the validity of the law can be demonstrated in the following table:

p	q	$p \rightarrow q$	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

Examples:

From "If a is less than zero, then a^2 is greater than zero," we may conclude that "If a^2 is not greater than zero, then a is not less than zero."

From "If the temperature remains above 0°C, then it will not snow," we may conclude that "If it snows, then the temperature has not remained above 0°C."

From $(r \vee \sim s) \rightarrow t$, we may conclude that $\sim t \rightarrow \sim (r \vee \sim s)$.

The *Law of Detachment* states that if a conditional is true and its antecedent is true, then its consequent is true. In symbols, $[p \wedge (p \rightarrow q)] \rightarrow q$.

Examples:

From "If today is Tuesday, then this is Belgium," and "Today is Tuesday," we may conclude that "This is Belgium."

From "If Mr. Clark is not in a good mood, then either his arthritis is acting up again or his daughter did not pass French," and "Mr. Clark is not in a good mood," we may conclude that "Either his arthritis is acting up again or his daughter did not pass French."

The reasonableness of the Law of Detachment can be demonstrated by appealing to the truth table for implication:

p	q	p → q
(T)	(T)	(T)
T	F	F
F	T	T
F	F	T

In row 1, it is the case that p → q is true, as indicated by the circled T, and also that the antecedent, p, is true. We note that in this row, which represents the only case in which the implication and its antecedent are both true, the consequent, q, is also true. Examples will help students see that the validity of the rule is unchanged for any replacement of p and with "real" sentences. Also, the following truth table may be completed, using the rules for Λ and →.

p	q	p → q	p Λ (p → q)	[p Λ (p → q)] → q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Thus, we see that the sentence, [p Λ (p → q)] → q, which is the statement of the Law of Detachment, is always true regardless of the individual truth values of p and q. It might be useful, in anticipation of future work, to symbolize the two examples given above in the following fashion:

Let T represent "Today is Tuesday"
Let B represent "This is Belgium"

Statements	Reasons
1. T → B	1. Given
2. T	2. Given
3. B	3. 1, 2, Law of Detachment

The *Rule of Disjunctive Inference* states that if a disjunction is true, and one of the disjuncts is false, then the other disjunct is true. In symbols: $[(p \vee q) \wedge \sim p] \rightarrow q$ and $[(p \vee q) \wedge \sim q] \rightarrow p$.

The validity of *Disjunctive Inference* can also be demonstrated by appealing to truth tables:

p	q	p V q	~p	(p V q) Λ ~p	[(p V q) Λ ~p] → q
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

Similarly, it can be shown that $[(p \vee q) \wedge \sim q] \rightarrow p$.

Examples:

From a = 0 or b = 0, and $a \neq 0$, we may conclude that b = 0.

From "Mr. Clark's arthritis is acting up again or his daughter did not pass French," and "His daughter passed French," we may conclude that "Mr. Clark's arthritis is acting up again."

The previous example may be presented in "statement-reason" form as follows:

Let A represent "Mr. Clark's arthritis is acting up again."
Let P represent "Mr. Clark's daughter passed French."

Statements	Reasons
1. A V ~P	1. Given
2. P	2. Given
3. A	3. 1, 2, Disjunctive Inference

This solution assumes, without statement, the Law of the Double Negative: $p \leftrightarrow \sim(\sim p)$, which some teachers might wish to consider separately.

The *Law of the Syllogism* is a rule of inference which involves three conditionals. This law is perhaps most easily stated in symbols: $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$.

Example 1:

From "If a man lives in Sioux City, then he lives in Iowa," and "If a man lives in Iowa, then he is a midwesterner," we may conclude that "If a man lives in Sioux City, then he is a midwesterner."

Example 2:

From If $3x + 5 = 7$, then $3x = 2$, and If $3x = 2$ then $x = \frac{2}{3}$, we may conclude that If $3x + 5 = 7$, then $x = \frac{2}{3}$.

Example 3:

From "If Tina is sick on Friday or if she is out of town, then Scott will ask Alyson to go canoeing with him," and "If Scott asks Alyson to go canoeing with him, Barbie will be fit to be tied," we may conclude that "If Tina is sick on Friday or if she is out of town, then Barbie will be fit to be tied."

As in the case of the other rules of inference discussed, the Law of the Syllogism can be justified with truth tables, although this time the table will contain eight rows, since this law involves three sentences.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	F	T	F	T	F	T	T
T	T	F	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	F	T	T	T	T	T

Since the final column shows all T's, we see that the rule is valid for any replacements of p, q, r.

The Law of the Syllogism can be used to provide an effective illustration of the essential distinction between the validity of a logical process and the truth of an individual premise involved in the process. Consider, for example, how, using valid reasoning, a true conclusion is drawn from false premises in the following:

Premises: If a man lives in Eau Claire, then he lives in Central Afghanistan.
If a man lives in Central Afghanistan, then he lives in Wisconsin.

Conclusion: If a man lives in Eau Claire, then he lives in Wisconsin.

Using the laws developed so far, examples like the following can be considered:

Either Fred or Sam murdered Mr. Green. If Sam murdered Mr. Green, then the butler was off Thursday night. If the butler was off Thursday night, then Nurse Partridge did not witness the crime. However, it is known that Nurse Partridge did witness the crime. Who murdered Mr. Green?

Sample solution:

Let F represent "Fred murdered Mr. Green."
Let S represent "Sam murdered Mr. Green."
Let B represent "The butler was off Thursday night."
Let P represent "Nurse Partridge witnessed the crime."

Statements	Reasons
1. $F \vee S$	1. Given
2. $S \rightarrow B$	2. Given
3. $B \rightarrow \sim P$	3. Given
4. $S \rightarrow \sim P$	4. 2, 3, Law of the Syllogism
5. $P \rightarrow \sim S$	5. 4, Law of Contraposition
6. P	6. Given
7. $\sim S$	7. 5, 6, Law of Detachment
8. F	8. 1, 7, Disjunctive Inference

That is to say, Fred is the guilty one. (Of course, other orders of inference are possible.)

Many students find problems like this enjoyable and challenging, while others find it difficult to "get the hang of them" or to "know where to start." Whether to introduce such problems, or how many to introduce, or how complex they should be, are decisions each teacher will have to make. Texts on logic, or with chapters on logic, can provide additional examples, and, of course, teachers can make up their own, for their own purposes.

De Morgan's Laws give rules for the negation of a conjunction and of a disjunction. They are:

(1) The negation of the conjunction of two statements is equivalent to the disjunction of the negation of each; $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$.

(2) The negation of the disjunction of two statements is the conjunction of the negation of each; $\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$.

Again, the statement of the rule in symbols may be more expressible than the statement in words. (This is not a defect, for often effective symbolism

is part of what mathematics is all about.) In any case, a few examples help to make the laws clear: If it is false that "Mary passed French and Sally passed Latin," then it must be true that "Mary did not pass French" or that "Sally did not pass Latin," or that neither girl passed either subject. That is,

original: Mary passed French and Sally passed Latin.

negation: Mary did not pass French or Sally did not pass Latin.

(Recall that or means either or both.)

Likewise,

original: Fred or Mary is a pharmacist.

negation: Fred is not a pharmacist and Mary is not a pharmacist.

(That is, Fred and Mary are not pharmacists.)

The rule whose Latin name is *Modus Tollens* states that if an implication is true and its consequent is false, then its antecedent is also false: $[(p \to q) \wedge \sim q] \to \sim p$ The reasonableness of this rule of denial of the consequent is most easily seen through the use of truth tables; the reasoning is the same as in the case of the Law of Detachment.

p	q	p → q
T	T	T
T	F	F
F	T	T
(F)	(F)	(T)

The case represented by row 4 is the only one where the implication is true and the consequence is false, and, as we see, in this case the antecedent is also false.

Actually, this rule is the same (has the same logical effect) as the Law of Detachment and the Law of Contraposition used successively. For, given (the truth of) $R \to S$ and $\sim S$ we may either conclude $\sim R$ directly using *Modus Tollens* or we may reason:

(i)	$R \to S$	(i)	given
(ii)	$(\sim S \to \sim R)$	(ii)	the Law of the Contrapositive
(iii)	$\sim S$	(iii)	given
(iv)	$\sim R$	(iv)	ii, iii, Law of Detachment

Modus Tollens, then, in terms of the development of this section, is a convenience rather than a necessity.

Example:

Given: "If I am going to the dance, I will buy new shoes."
and "I am not going to buy new shoes."

We may use *Modus Tollens* to conclude directly.

"I am not going to the dance."

In some logic proofs, teachers may wish to make use of the Law of Simplification. This law is the formal justification for the dissolution of a conjunctive. In symbols: $(p \wedge q) \rightarrow p$ and $(p \wedge q) \rightarrow q$. In addition, some teachers may wish to refer to the Law of Substitution, which permits, in a manner familiar from algebraic and geometric proofs, replacement of any statement by another to which it is logically equivalent.

Some teachers may wish to consider the method of proof known as indirect proof. In an indirect proof, the negation of the statement to be proved is assumed to be true, and it is then shown that this assumption leads, by a series of otherwise correct procedures, to a statement known to be false. Usually, this false statement is a contradiction — a statement of the form $(q \wedge \sim q)$ — which is necessarily false. Now, since all other procedures in the proof were correct, it must have been the original assumption that led to the contradiction, and, therefore, the assumption is false. If the assumption is false, its negation — the original statement — is true. This process, sometimes called the Principle of Non-Contradiction, may by symbolized in this way: $[p \rightarrow (q \wedge \sim q)] \rightarrow \sim p$.

Example:

Assume in the case of the murder of Mr. Green that it is suspected that Fred is the culprit. Following is one version of an indirect argument that Fred is guilty. It begins by assuming that he is not guilty.

Statements	Reasons
1. $\sim F$	1. Assumption
2. $F \vee S$	2. Given
3. S	3. 1, 2, law of Disjunctive Inference
4. $S \rightarrow B$	4. Given
5. $B \rightarrow \sim P$	5. Given
6. $S \rightarrow \sim P$	6. 4, 5, law of Syllogism
7. $P \rightarrow \sim S$	7. 6, law of Contrapositive
8. P	8. Given
9. $\sim S$	9. 7, 8, law of Detachment
10. $S \wedge \sim S$	10. 3, 9, definition of conjunction
11. F	11. 1, 10, Principle of Noncontradiction; $[p \rightarrow (q \wedge \sim q)] \rightarrow \sim p$

Teachers may wish to point out at this time, or earlier, that the rules studied in this section are examples of *tautologies;* that is, of statements which, because of their structure, rather than their content, are always true. One of the easiest of these to understand has been met before: $p \vee \sim p$. It might be mentioned that in nonmathematical discussion, tautological reasoning is a technique looked on with disfavor: to describe the arguments of a politician or social scientist as tautological is no compliment. Such a description implies that the person has advanced no substantial argument, but has just structured it in such a way that the desired conclusion is forced from this structure.

Teachers may also wish to investigate the writings on logic (including some very amusing examples) of Lewis Carroll, known to everyone as the author of *Alice in Wonderland* and *Through the Looking-Glass*. His real name was Charles Lutwidge Dodgson, and he was professor of mathematics, one of whose specialties was logic.

C. An open sentence contains a variable and may assume different truth values as the variable changes or may have the same truth value for all replacements of the variable. A statement that an open sentence has the same truth value for all replacements of the variable is said to be universally quantified and may be expressed using a universal quantifier such as All, For all x, every, or the symbol $\forall_x$. A statement which is meant to apply to at least one (possibly every) replacement of the variable is said to be existentially quantified and may be expressed using an existential quantifier such as Some, For some x, There exists an x such that, At least one, or the symbol $\exists_x$.

Examples of statements which are universally quantified:

(i) For all x, $x + x = 2x$

(ii) $\forall_x$ $x + x = 2x$ (equivalent to i)

(iii) All rectangles are parallelograms.

(iv) $\forall_{x \in Z}$ $x + 0 = x$, where Z = {integers}

(v) $\forall_{x \in Z}$ $x + 2 = x$. (This universally quantified statement is false.)

(vi) No elephant is green. (That is, All elephants are not green.)

(vii) None of the powers of 2 is divisible by 5. (That is, $\forall_{x \in T}$ 5 does not divide x, where T = {powers of 2}.)

Examples of statements which are existentially quantified:

(i) Some integers are divisible by 3.

(ii) There exists an integer x such that x is divisible by 3.

(iii) $\exists_{x \in Z}$ x is divisible by 3.

(iv) At least one integer is divisible by 3.
(i, ii, iii, iv are equivalent.)

(v) Some equilateral triangles are polygons. (This existentially quantified statement happens to be true for all elements of the replacement set.)

(vi) Not all animals are ocelots. (That is, there exists an animal which is not an ocelot.)

To show that a universally quantified statement is false, it is sufficient to find one case (a counterexample) for which the original statement is false.

Example:

For all x, $x^2 > 0$ is not true because if $x = 0$, $x^2 \not> 0$. Since there is one x (i.e., 0) for which the statement is false, it cannot be true for all x.

Put another way:

Original: $\forall_x\ x^2 > 0$

Negation: $\exists_x\ x^2 \not> 0$

To show that an existentially quantified statement is false, it is necessary to demonstrate that there are no cases for which the original statement is true. For example, if we wish to prove that the statement, "Some elephants can solve quadratic equations," is false, we need to demonstrate that all elephants cannot solve quadratic equations (that is, that no elephant can solve quadratic equations).

Note that the negation of a universally quantified statement is an existentially quantified statement, and that the negation of an existentially quantified statement is a universally quantified statement.

Teachers are urged to take special care to distinguish between statements of the form "All x is not y" and "Not all x is y." For example, the statements "All men are not honest" and "Not all men are honest" clearly do not assert the same idea.

II. Mathematical Systems

A. Students have been developing familiarity with mathematical systems — the real numbers and its subsets — since their initial experiences in arithmetic. It is hoped that by introducing some finite systems at this point, the student may be helped to recognize the significance of those properties which underlie and characterize the systems most frequently encountered.

In a finite system, it is possible to define a binary operation on the elements of a set simply by exhibiting the result of that operation for each pair of elements.

Example 1:

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

This table defines the operation * for each pair of elements in S = {a, b, c}. From the table, it can be seen, for example, that b * a = b and that c * c = b. Also, this system has two important mathematical properties: it has an identity element, a, and each element has an inverse. The inverse of a is itself, a, (notation: $a^{-1} = a$); and b and c are inverses of each other (i.e., $b^{-1} = c$ and $c^{-1} = b$).

Example 2:

#	a	b	c
a	b	a	c
b	c	a	b
c	a	c	b

This table also defines an operation, #, on each pair of elements in S = {a, b, c}. From the table it can be seen that while each pair has a unique image in S under the operation # (i.e., {a, b, c} is closed under #), other common mathematical properties are missing. There is no identity element. Since there is no identity, the notion of inverse has no meaning in this system. The system is not commutative (a # c = c but c # a = a) and it is not associative [(a # b) # c = c but a # (b # c) = a].

This approach however may seem arbitrary and contrived to students so systems whose introduction can be justified by some relationship to an everyday situation are often more useful. Perhaps a good choice is "clock arithmetic." Students may begin by considering the set of integers 1 through 12, or some smaller system, and the operation "addition" as it occurs on a clock.

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	8	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
.												
.	etc.											
.												

By use of the table, or appeal to the clock, it can be seen that in this system, "addition" is closed, commutative and associative; that 12 is the identity for the system (12 + 2 = 2, 10 + 12 = 10 etc.) and, since 12 behaves like the identity in the reals (zero), may be renamed 0 if we wish; that every element has an inverse (2 and 10 are inverses because 2 + 10 = 10 + 2 = 12 or 0, the identity). The calendar week suggests a "clock" of seven elements which can be designated 0 through 6 with 0 the identity element. A "clock" of any number of elements provides no new concept under the operation addition.

However, if we consider a "clock" of 12 elements and a "clock" of 7 elements under the operation "multiplication," we find significant differences in the two resulting tables. In both cases, the operation is closed, commutative and associative and there is an identity, 1. But the behavior of 0 deserves special consideration. In the system of 7 elements, $0 \times a = 0$ for all a, and, as in most systems familiar to the students, if $a \times b = 0$ then $a = 0$ or $b = 0$. In the system of 12 elements, $0 \times a = 0$ for all a but $a \times b = 0$ does <u>not</u> require that $a = 0$ or $b = 0$: for example, $2 \times 6 = 0$, $8 \times 3 = 0$. Also, in the system with 7 elements, there is an inverse for each non-zero element; this is not the case in the 12 element system. Students should investigate as many different systems as necessary to recognize significant differences or similarities.

It is hoped that the understanding of identity, commutativity, associativity, etc., can be achieved before formal definition and that definitions can come from students after they become familiar with examples of those concepts.

Teachers may wish to introduce the following examples of finite systems of two elements and discuss the correspondence between these systems and the clock 2 arithmetic:

+	Even	Odd
Even	Even	Odd
Odd	Odd	Even

x	Even	Odd
Even	Even	Even
Odd	Even	Odd

The system described here as "clock" arithmetic can, of course, be presented to more mathematically able students as congruence classes (classes of remainders after division by an integer n) or can be designated $(\not{Z}n, +)$ and $(\not{Z}n, \times)$ if such terminology or symbolism seems more appropriate and does not obscure the basic investigation involved.

Simple equations such as $5 + x = 2$ or $5x = 1$ or $x^2 = 1$ can be solved in a given system by appealing to the addition or multiplication tables or to the usual equation-solving techniques involving additive and multiplicative inverses where they exist. In some cases the solution set may be empty.

It is recommended that examples such as the following be considered by most classes.

If $5 + x = 2$
then $x = 4$ for clock 7
$x = 9$ for clock 12
$x = 5$ for clock 8

If $5x = 1$
then $x = 3$ for clock 7
$x = 5$ for clock 12

There is no solution for clock 10 (i.e., the solution set is the empty set).

If $x^2 = 1$
then $x = 1$ or $x = 6$ for clock 7
$x = 1$ or $x = 11$ for clock 12
$x = 1$ or $x = 7$ for clock 8

(Notice that in each case, the solution of $x^2 = 1$ is 1 and its additive inverse, just as in the reals.)

B. The experience the student has had in recognizing properties of a mathematical system should lead to the definition of *group*. A *group* is a set together with a binary operation such that:

(1) the set is closed under the binary operation
(2) the operation is associative
(3) there is an identity element under that operation
(4) each element has an inverse under that operation

Universal and existential quantifiers may be used to introduce the definitions of a group. It can be verified that the following systems satisfy this definition, and are thus groups.

(i) integers, rationals, or reals, under addition
(ii) positive rationals or positive reals, under multiplication
(iii) even integers under addition
(iv) the set S, under operation * (see example 2, section A, above)
(v) $\mathbb{Z}_n$, under addition, for any natural number n
(vi) Any set with exactly one element 🌳 under operation 🌲 defined in such a way that 🌳 🌲 🌳 = 🌳

🌲	🌳
🌳	🌳

(vii) The nonzero (i.e., nonadditive identity) elements of clock n, where n is prime, under multiplication. On the next page is the table for n = 5

x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(viii) The set T = {2, 4, 6, 8} under the operation ⊗ defined such that the product of any two elements is the units digit of the product under ordinary multiplication.

⊗	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

(Note that for this group the identity element is 6.)

A group need not be commutative although most examples within students' experience, including all those given above, are commutative. Teachers may wish to demonstrate the existence of a noncommutative group by presenting the following permutation group.

Label three chairs (or three students) 1, 2, 3 in order from left to right. Let a new arrangement of chairs be defined by

$$a \begin{vmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{vmatrix}$$

$$b \begin{vmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{vmatrix} \qquad c \begin{vmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{vmatrix} \qquad d \begin{vmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{vmatrix} \qquad e \begin{vmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{vmatrix} \qquad i \begin{vmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{vmatrix}$$

Let a * b consist of doing a first then b so that

$$a \begin{vmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{vmatrix} \qquad b \begin{vmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{vmatrix}$$

a * b is the same as $\begin{vmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{vmatrix}$ which is c.

The class may then be asked to complete the following table:

*	i	a	b	c	d	e
i						
a			c			
b		d				
c						
d						
e						

Note that b * a consists of doing

1	2	3					
↓	↓	↓	b				
2	1	3		or	↓	↓	↓
↓	↓	↓	a		2	3	1
2	3	1					

thus yielding d. Therefore, a * b ≠ b * a.

When completed, it can be shown that this table defines a group which is not commutative. (This group is the permutation group on three elements, which is equivalent to the group of the symmetries of an equilateral triangle. Teachers may choose to consider the table from either of these points of view. The group of the symmetries of the square is another example of a noncommutative group.)

It should be noted that there exist no noncommutative groups with fewer than 6 elements.

The following are examples of systems which fail to be groups:

(ix) whole numbers (non-negative integers), under addition (inverse property fails)
(x) integers under multiplication (inverse property fails)
(xi) rationals or reals under multiplication (zero has no inverse). (Note, however, that the nonzero rationals or the nonzero reals, under multiplication, do form a group.)
(xii) the set S under the operation # (see example 2, section A)

Students should be able to show that a set with a given operation is or is not a group by demonstrating that it satisfies or fails to satisfy the group definition.

Teachers may wish to use the following notation in their discussions of groups: If a set G and an operation * form a group, this group may be designated (G, *). Thus, if $\mathbb{Z}$ is the symbol used to represent the set of integers, then ($\mathbb{Z}$, +) denotes the group of the set of integers with addition.

C. When equations in finite mathematical systems were first introduced, their solution may have been arrived at by reference to the table which defines the system. An investigation of basic laws needed to solve equations can point up those properties of a system which are essential to make the system "work." Since the result of an operation is unique, a = b implies that ac = bc, since if a and b are the same element (even if the symbols for this element are different), the result of operating with c must necessarily be the same element. The converse of this statement, the cancellation law (ac = bc implies a = b) can be shown to be true in any system which satisfies the definition of a group, as the following demonstration show.

If	$ac = bc$	
	c^{-1} exists	by inverse property
then	$(ac)c^{-1} = (bc)c^{-1}$	by the uniqueness property
	$a(cc^{-1}) = b(cc^{-1})$	by the associative property
	$a \cdot 1 = b \cdot 1$	by the definition of *inverse*
	$a = b$	by the definition of *identity*

Teachers should note that in the preceding discussion, symbolism associated with the operation of multiplication (including the identity element) has been used. The proof is valid, of course, in any system where the indicated properties are present, regardless of the operation (or its symbol) involved, and some teachers may wish to present similar proofs for other systems.

The Cancellation Law, then, holds in any system which can be shown to be a group. In particular, it will hold in all of the additive clock arithmetic systems, and it can be verified that it will hold as well for the nonzero elements of a multiplication clock of n elements, if n is prime. The law is also valid in the multiplicative groups of nonzero rationals and nonzero reals, a fact expressed in its usual statement for these systems: If $ac = bc$ and $c \neq 0$, $a = b$. A more formal demonstration of the validity of the usual method of solving linear equations such as $5 + x = 2$ may include the following proof of:

In a group (G, o), $a \circ x = b \rightarrow x = a^{-1} \circ b$

If	$a \circ x = b$	
then	a^{-1} exists	since every element of G has an inverse
and	$a^{-1} \circ (a \circ x) = a^{-1} \circ b$	by the uniqueness property
	$(a^{-1} \circ a) \circ x = a^{-1} \circ b$	by the associative property
	$1 \circ x = a^{-1} \circ b$	by the definition of *inverse*
	$x = a^{-1} \circ b$	by the definition of *identity*

In the linear equation $5 + x = 2$, 5 corresponds to a, + to o, and 2 to b. This proof does not show that $a^{-1} \circ b$ is the only solution of $a \circ x = b$; demonstration of this fact should be reserved for more capable students. The level of abstraction and of rigor presented in these notes is most likely appropriate to the more mathematically able student, and teachers may find a less formal approach more suitable for many of the students they teach.

D. The importance of the distributive property lies in the fact that it links two operations, usually called addition and multiplication.

With this property it is possible to assemble all of the necessary conditions for a *field*. The reintroduction of the distributive property at this time can be used to review some of the basic algebraic concepts of Course I.

A field is a set together with two operations such that:

(i) the set forms a commutative group with identity zero under the first operation
(ii) nonzero elements form a commutative group under the second operation
(iii) the distributive property unites the two operations (the second distributes over the first)

Teachers may wish to use the notation (F, +, ·) to denote the field formed by set F defined under the operations + and ·.

Examples:

(1) If n is a prime, the clock arithmetic with n elements with addition and multiplication forms a field. If n is composite, then the set will have at least one nonzero element without a multiplicative inverse, and hence condition (iii) will fail.
(2) The rationals and reals, each under addition and multiplication form fields. The integers under addition and multiplication do not form fields since the nonzero integers are not a group under multiplication. See section B, example x.
(3) It can be verified that the set K = {0, 2, 4, 6, 8} under the operations $\oplus$ and $\otimes$, where $\oplus$ and $\otimes$ are defined so that the result of each operation is the units digit of the result obtained under ordinary addition and multiplication, respectively, forms a field. (See section C, example viii.)

Students should be able to demonstrate whether or not a set with two operations forms a field. The teacher should point, as an example of the importance of fields, that the solution of linear equations, studied in Course I, is dependent on field properties.

Another important application of the study of fields is a theoretical justification of the solution of quadratic equations by factoring: that ab = 0 implies a = 0 or b = 0 is a statement which can be proved true in any field.

III. Euclidean Geometry

A. Many of the basic terms and concepts of geometry have been considered in the study preceding this course. The student should understand that many conclusions were reached through measurement and observation and that these conclusions will now be investigated more formally. This section will be directed toward helping students organize lines of thought into more "formal" "proofs" as well as toward learning

new geometric relationships. Some teachers may use a transformation approach to the material of this section; it should be emphasized that no specific approach is mandated by the syllabus. Teachers should use whatever point of view appears to them to be most appropriate.

Since every mathematical statement must be expressed in words, symbols, or statements previously defined or agreed upon, it is impossible to define all terms or prove all statements. Therefore it is necessary to choose some terms whose meanings are accepted without definition, and some assumptions whose truth is accepted without proof as the basis of a logical system. It is not intended that any attempt be made to keep the list of undefined terms and postulates at any strict minimum, and teachers may wish to postulate some statements which could be proved as theorems when the proof does not appear to be appropriate to their students.

B. The sequence of theorems presented here has been selected as the core of the section on Euclidean geometry because it contains theorems of wide applicability including adaptability to other approaches to geometry, presents a basis for other theorems simple enough to be proved as originals by most students, and culminates in a proof of the Pythagorean theorem.

The list is intended to be skeletal, and it is expected that the teacher will necessarily present assumptions, definitions and ancillary theorems as needed in the development of the sequence. For example, before discussing the theorem which deals with the relationship of an exterior angle of a triangle to the nonadjacent interior angles, it will be necessary to introduce some concept of betweenness. This concept is important, not only in itself, but also in that it serves as a good example of one fact — that in mathematics many things which may appear initially to be "obvious" or "trivial" are not obvious or trivial at all, and are, often in fact, worthy of considerable attention. In addition, teachers may also find it suitable at this time to discuss with their classes the potential for error involved in relying too heavily on those diagrams which have been prepared only as an aid in the solution of a problem. Texts on Euclidean geometry vary in the presentation of such topics and it is assumed that teachers will provide a logically sound development consistent with their students' ability.

Among the conventional assumptions presented here are the assumptions that two triangles are congruent if the following parts are respectively congruent: (1) two angles and the included side, (2) two sides and the included angle, and (3) three sides. If students have not had enough experience with congruence in preceding courses to accept these assumptions as plausible, this experience can be provided through constructions and the use of physical models.

The concept of parallelism leads to many theorems with relatively simple proofs which are in themselves useful facts and which could provide opportunities for original proof. Some examples are:

(1) A diagonal of a parallelogram separates the parallelogram into two congruent triangles.
(2) The opposite sides of a parallelogram are congruent.

(3) The diagonals of a parallelogram bisect each other.
(4) If the opposite sides of a quadrilateral are congruent, the figure is a parallelogram.
(5) If two sides of a quadrilateral are congruent and parallel, the figure is a parallelogram.
(6) If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.
(7) The diagonals of a rhombus are perpendicular to each other.
(8) The segments joining the midpoints of consecutive sides of a quadrilateral form a parallelogram.

Any standard geometry text will supply similar statements.

The introduction of similar triangles, like that of parallelism, admits of a variety of approaches. Often the statement on similar triangles suggested here as an assumption is *proved* after the introduction of some other assumption which seems to be "more obvious," (for example, for any positive real number k, and any given Δ ABC, there exists a triangle similar to Δ ABC with constant of proportionality k.) In any case, if an approach is adopted which involves a "proof" valid only for the "commensurable case," individual teachers will want to decide what stress to give a discussion of its limitations and the need for some assumption to bridge the gap to the "noncommensurable case."

Many applications of the theorems on similar triangles arise from practical situations (maps, scale drawings, etc.) and provide numerical exercises useful in the review and extension of algebraic skills.

Teachers who wish to introduce geometric inequalities may wish to consider the following sequence of theorems:

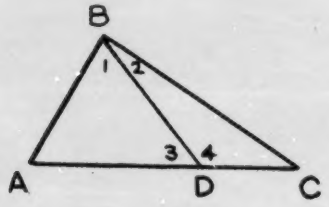
- The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle.
- If two angles of a triangle are unequal in measure, the sides opposite them are unequal in length and the longer side is opposite the larger angle.
- If two sides of a triangle are unequal in length, the angles opposite them are unequal in measure and the larger angle is opposite the longer side.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Following are two examples typical of those which teachers may wish to consider.

Example 1:

Two sides of a triangle have lengths 2 and 5. Find all possible lengths of the third side.

Example 2:



Given: AD > AB

Prove: m ∡ 1 > m ∡ C

Many geometric statements involving inequalities can be proved by means of an indirect proof and may serve to reinforce some of the logical considerations of section 1.

C. The use of compasses and straightedge has been introduced in previous courses. The constructions suggested should be accompanied by a proof or explanation of their validity although this aspect of the topic should not be overstressed if so doing subtracts from students' enjoyment of the topic. Formal proofs may be considered as a consequence of the theorems developed. It is hoped that students, especially those whose thinking tends toward the visual, can be encouraged to enjoy the reward of a well-constructed figure. Teachers may wish to discuss briefly the topic of classic impossible constructions (e.g., the trisection of an angle with compass and straight edge.)

D. The fact that parallelism, similarity, congruence, as well as equality, are some of the most important, frequently used, and readily accessible examples of an *equivalence relation* suggests study of this kind of relation at this time. It occurs so frequently in different areas of mathematical study that some teachers, depending on the ability of their classes, may wish to give attention to this unifying concept at this time.

An equivalence relation is a *relation* which is *reflexive*, *symmetric*, and *transitive*.

The term, relation, may be considered in several ways. One is simply to define relation as a (any) set of ordered pairs. Thus,

{(1,2), (2,1), (2,3)}
{...(4,-2), (1,-1), (0,0), (1,1)...}
{(Alabama, Montgomery), (Alaska, Juneau), ... (Wyoming, Cheyenne)}

are relations. Or we may specify some rule or method of assignment of one set of entities (these will correspond the set of first components in each set of ordered pairs) to a second set (corresponding to the second components). Thus, "lives in the same city as," "is the father of," "is greater than," "is a subset of," "is four less than," "is the square of," "is congruent to," etc., are all relations. Very often we deal with a relation from a set A to itself.

Example 1:

Let A = {states in the U.S.} and R = "is greater in area than." on set A. Then, R = {(Alaska, Alabama), (Alaska, Arkansas), (Alaska, Colorado), ... (Alaska, Texas) ... (Colorado, Delaware), ... (North Carolina, South Carolina), ... (Delaware, Rhode Island)}. There is no $x \in A$ such that (Rhode Island, x) $\in$ R or no $x \in A$ such that (x, Alaska) $\in$ R.

Example 2:

The relation S "is the square of" on the set of integers, is {...(9, -3), (4, -2), (1, -1), (0,0), (1,1)...}.

Example 3:

The relation "equals" on the set of reals is the set of all ordered pairs whose first and second components are the same.

Example 4:

If F is the relation "is the father of" on M = {New Jersey males} then (a, b) $\in$F means that a is b's father for all a, b $\in$ M.

Example 5:

If S "is congruent to" on the set T of all triangles then for a, b $\in$ T, (a, b) $\in$ S means $\Delta a \cong \Delta b$.

A relation R on set A is *reflexive* if for all $a \in A$, (a, a) is an element of R.

Examples of relations which are reflexive:

(1) The relation "=" (equality) on the set of reals, or any subset of the reals, since, for any real number r, r = r.

(2) The relation R = {(1, 1), (1, 2), (2, 2)} on B = {1, 2}, since for all $x \in B$, (x, x) $\in$ R.

(3) The relation "is similar to" on the set of all triangles or any subset of that set.

(4) The relation "is parallel to" on the set of lines in a plane (assuming that a line is defined to be parallel to itself).

Examples of relations which are not reflexive:

(1) The relation "<" on the reals of any subset of the reals. (It can never be the case that for all x in such a set, $x < x$.)

(2) The relations "is a proper subset of," "is taller than," "earns more money than," etc., on appropriate sets.

(3) The relation R = {(1, 2), (2, 2)} on B = {1, 2} since there exists $x \in B$ (namely 1) which does not stand in relation R to itself, i.e., $(1, 1) \in R$.

A relation R on set A is *symmetric* if for all $a, b \in A$, $(a, b) \in R \rightarrow (b, a) \in R$). Note that this definition requires that $(b, a) \in R$. when $(a, b) \in R$. Examples of relations which are symmetric:

(1) The relation "=" on the reals or any subset of the reals since for all a, b ∈ {reals} if a = b then b = a.

(2) The relations $\cong$, $\sim$ on the set P of all polygons or any subset of P.

(3) The relation R = {(1, 2) (2, 1) (4, 1) (1, 4)} on A = {1, 2, 3, 4}. Note that (3, 1) needn't be in R since (1, 3) isn't.

(4) The relation "lives in the same state as" on the set A of Americans, or any subset of A.

Examples of relations which are not symmetric:

(1) The relation "<" on the reals or any subset of the reals since it will never be the case that for all x, y in such a subset, $x < y \rightarrow y < x$.

(2) The relation "$\leq$" on the reals or any subset of the reals with more than one element.

(3) The relations "is taller than," "owns more French books than," "is the cube of," etc., on appropriate sets.

A relation R on set A is *transitive* if for all a, b, c ∈ A, $(a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$.

Examples of relations which are transitive:

(1) The relation "has the same weight as" on the set of all people, or any subset of that set. (Certainly, if A has the same weight as B and B has the same weight as C, then A has the same weight as C.)

(2) The relations "=" "<" "$\leq$" on the set of reals or any subset; the relation "$\subset$" on a set of sets.

(3) The relations "$\cong$" "$\sim$" on the set of geometric figures or any subset.

(4) The relation {(1, 2), (2, 3), (1, 1), (1, 3), (2, 1), (2, 2)} on {1, 2, 3}.

Examples of relations which are not transitive:

(1) The relation "sits next to" on the set of students in a classroom.

(2) "Differs from by exactly 5" on the set of reals or any subset.

Readers should verify that equality, congruence, similarity, and parallelism are each reflexive, symmetric, and transitive and are therefore equivalence relations.

Note: The notation aRb is often used for $(a, b) \in R$. Thus, in the first example given above xRy would indicate that state x is greater in area than state y.

IV. Analytic Geometry

A. A review of the basic elements and vocabulary of a rectangular system of coordinates will strengthen the student's awareness of the one-to-one correspondence between the points of the plane and the ordered pairs of real numbers. This correspondence makes it possible to study geometric concepts in terms of number and apply the techniques of algebra to the study of geometry. Many analytic proofs developed in this section depend on the theorems concerning parallel and perpendicular lines and similar triangles developed in section III.

B. Having chosen for the plane a coordinate system in which the axes are perpendicular to each other, it is possible to use the Pythagorean theorem to derive a formula for the distance between two points in this plane. The development should lead to a demonstration which is independent of the quadrant in which the points lie. Students should be able to use their understanding of distance in exercises such as the following:

(i) Show that the triangle whose vertices are (-3, 0), (1, -2), and (-1, 4) is isosceles. Show that the altitude to the base separates the triangle into two congruent, isosceles triangles.

(ii) Plot the points (-1, 1) (2, 5), (2, 10), and (-1, 6). Show that the figure formed by joining the points in order is a rhombus.

(iii) Write the equation of the set of points equidistant from (1, 1) and (5, 3).

The location of the midpoint of a line segment, or a point which divides a line segment into parts whose lengths are in a given ratio, follows from the assumption made in section III that if a line is parallel to one side of a triangle and intersects the other two sides, it divides the other two sides into segments whose lengths are proportional. That $\left(\frac{x_1 + y_2}{2}, \frac{y_1 + y_2}{2}\right)$ are the coordinates of the midpoint of the line segment joining (x_1, y_1) and (x_2, y_2) can be verified using the distance formula. In some classes, student interest may suggest that this idea be extended to prove that the point $\left(\frac{bx_1 + ax_2}{a + b}, \frac{by_1 + ay_2}{a + b}\right)$ is the point which separates the line segment from (x_1, y_1) to (x_2, y_2) into segments the ratio of whose lengths is a:b.

C. *Slope* should be defined and used to prove the following:

(i) Two nonvertical lines in the coordinate plane are parallel if and only if their slopes are equal.

(ii) Two nonvertical lines in the coordinate plane are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other (i.e., the product of their slopes is -1).

The student should be able to relate the concept of slope to examples such as the following:

(a) Plot the points A(-1, 1), B(2, 5), C(2, 10), and D(-1, 6). Prove that the figure ABCD is a parallelogram. Prove that the diagonals of this parallelogram are perpendicular to each other. What conclusions can you draw about this parallelogram?

(b) Use slope to determine whether the following points are collinear: (-3, 1), (-1, 0), and (3, -2).

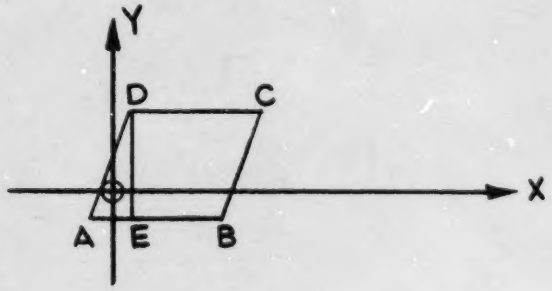
(c) Is the triangle whose vertices are (-3, 0), (1, -2), and (5, 6) a right triangle?

D. A thorough understanding of the concept of slope should enable the student to write an equation of a line given the slope and one point or given two points of the line. The idea of y-intercept should be discussed and the student should recognize $y = mx + b$ as an equation of the line through (0, b) with slope m.

While students should know the formulas for the areas of common plane figures (circle, triangle, parallelogram, etc.), the problem of determining the areas of specific geometric figures provides excellent applications of the foregoing material.

Examples:

(i) Find the area of parallelogram ABCD if A = (-1, -1), B = (6, -1), C = (8, 5), and D = (1, 5)



(AB = 7) (DE = 6) (Area = 42)

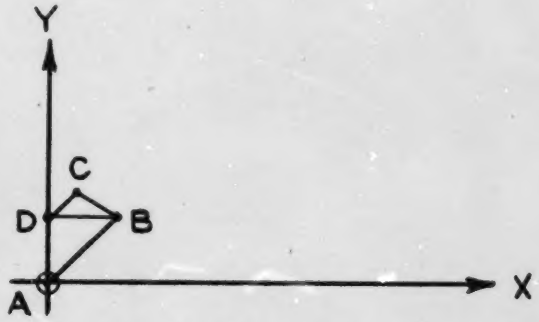
(ii) What is the area of the trapezoid ABCD if A = (0, 0), B = (3, 3), C = (1, 4), and D = (0, 3)?

Possible solutions:

(i) Diagonal $\overline{BD}$ separates the trapezoid into two triangles.

Area ABCD = Area ABD + Area DBC

$$= \tfrac{1}{2}(3)(3) + \tfrac{1}{2}(3)(1)$$
$$= \frac{9}{2} + \frac{3}{2}$$
$$= 6$$



(ii) Area ABCD = $\frac{1}{2}h(b + b')$

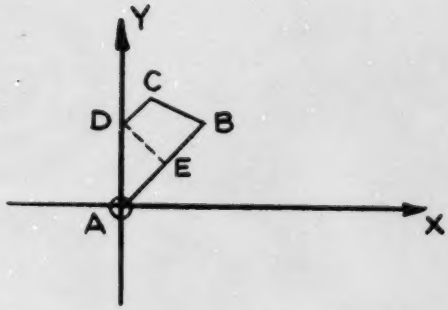
Let $\overline{DE}$ be the altitude from D to $\overleftrightarrow{AB}$

Equation of $\overleftrightarrow{AB}$: $y = x$

Equation of $\overleftrightarrow{DE}$: $y = -x + 3$

Coordinates of E: $\left(\frac{3}{2}, \frac{3}{2}\right)$

(simultaneous solution of $y = x$ and $y = x + 3$)



$$DE = \sqrt{(0 - \tfrac{3}{2})^2 + (3 - \tfrac{3}{2})^2} = \sqrt{\frac{9}{4} + \frac{9}{4}}$$

$$= \frac{3\sqrt{2}}{2}$$

$$AB = 3\sqrt{2}$$

$$CD = \sqrt{2}$$

$$\text{Area ABCD} = \frac{1}{2}\left(\frac{3\sqrt{2}}{2}\right)(3\sqrt{2} + \sqrt{2}) = \frac{1}{2}\left(\frac{3\sqrt{2}}{2}\right)(4\sqrt{2}) = 6$$

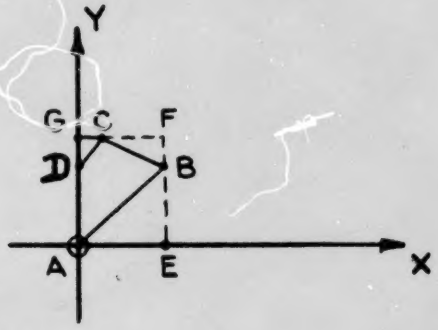
(iii) Enclose the figure in a rectangle AEFG
A = (0, 0), E = (3, 0), F = (3, 4), G = (0, 4)

Area ABCD = Area AEFG - Area AE[illegible] - Area CBF - Area DCG

$$= 12 - \frac{9}{2} - \frac{2}{2} - \frac{1}{2}$$

$$= 12 - \frac{12}{2}$$

$$= 6$$



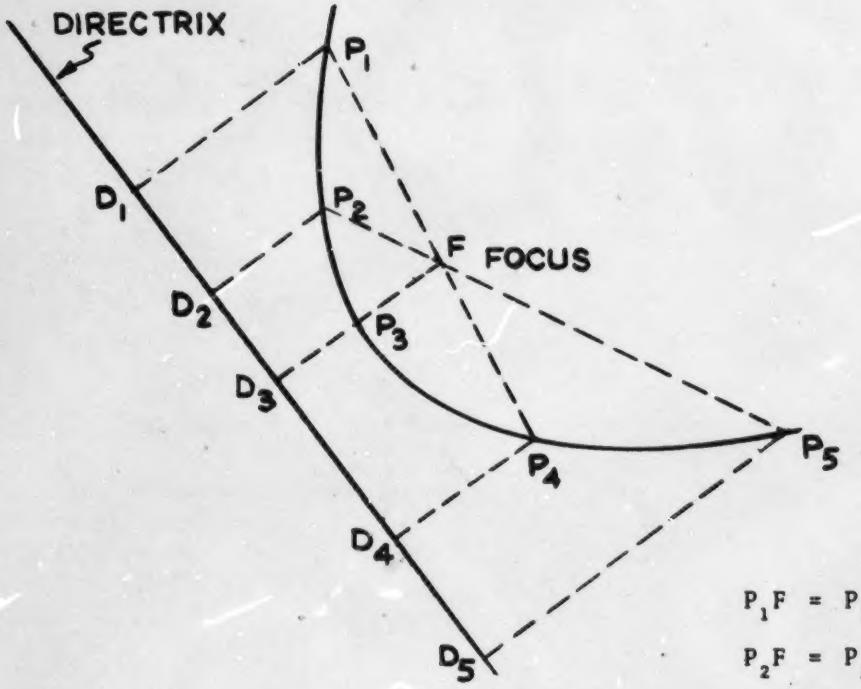
E. There have been many situations in which the student has considered "the set of all points such that ..." and should now to be equipped to deal with *locus* problems. The equation of the circle should be considered, with any given point as center.

Example 1:

Write the equation of a circle with center at (2, -1) and radius 5.

Using the distance formula and the Pythagorean theorem, we have, $(x - 2)^2 + (y + 1)^2 = 25$.

The parabola, the locus of points equidistant from a fixed point (focus) and a fixed line (directrix), provides an example of a relatively simple locus which is not obvious. For classroom study, the directrix should be parallel to one of the coordinate axes although the focus need not be on an axis. See diagram on the following page.



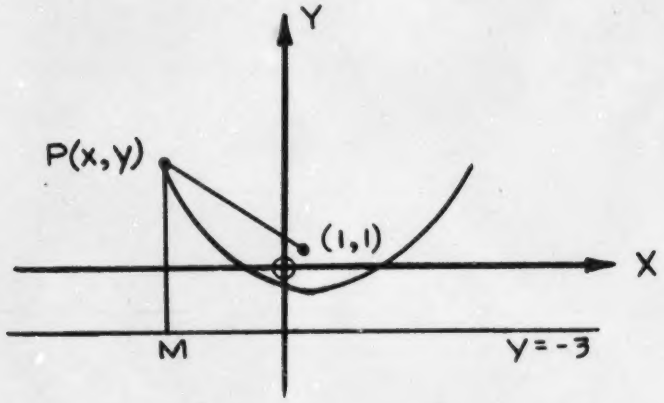
$P_1F = P_1D_1$

$P_2F = P_2D_2$

etc.

Example 2:

Write the equation of the parabola with focus at (1, 1) and directrix, $y = -3$.



$$PF = PM$$

$$\sqrt{(x - 1)^2 + (y - 1)^2} = y + 3$$

$$(x - 1)^2 + (y - 1)^2 = (y + 3)^2$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 = y^2 + 6y + 9$$

$$x^2 - 2x - 7 = 8y$$

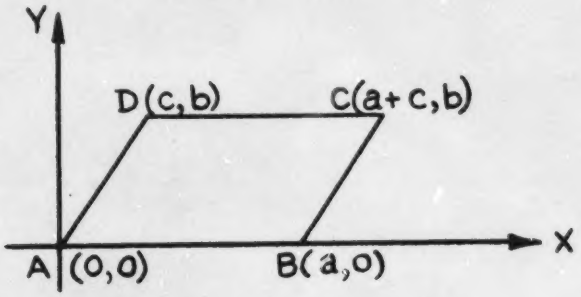
$$\text{or } y = \frac{1}{8}x^2 - \frac{1}{4}x - \frac{7}{8}$$

It is recommended that students be asked to draw several such loci, to determine their equations, and to see if they can discover, by comparing these equations to others they have seen in previous work, that this locus is a parabola, before being "officially told" that it is.

F. Each student, to the extent of his or her ability, should enjoy the discovery of geometric facts and relationships and should be encouraged to search for different approaches to familiar ideas. Coordinate geometry furnishes a new technique, often a simple one, for geometric proof. It is generally possible, in setting up such proofs, to choose the position of the x and y axes in a way which simplifies computation; such a choice in no way detracts from the universality of the proof. Some students will learn to simplify computation by such choices and should be encouraged, by comparing solutions, to do so. Most of the theorems on quadrilaterals listed in the notes for section III, part B, lend themselves to coordinate methods. Those which concern parallel or perpendicular lines, midpoints, or congruent line segments make use of the ideas of slope and distance which have been developed in this section. Teachers should note that at this level the student is equipped to deal analytically with congruent angles only if these angles are formed by parallel lines and a transversal.

Example 1:

Prove: The diagonals of a parallelogram bisect each other and the opposite sides are congruent.



ABCD is a $\square$

$\therefore \overline{AB} \parallel \overline{DC}$; slope of $\overline{AB} = 0$; slope of $\overline{DC} = 0 \rightarrow$ y coordinate of C = b

$\overline{AD} \parallel \overline{BC}$; slope of $\overline{AD} = \frac{b}{c}$ slope of $\overline{BC} = \frac{b}{c} \rightarrow$ x coordinate of C = a + c

Midpoint of $\overline{AC}$: $\frac{a + c}{2}, \frac{b}{2}$

Midpoint of BD: $\frac{a + c}{2}, \frac{b}{2}$

$\therefore$ the diagonals of $\square$ ABCD bisect each other

$AB = a$

$DC = a + c - c = a$

$\therefore \overline{AB} \cong \overline{DC}$

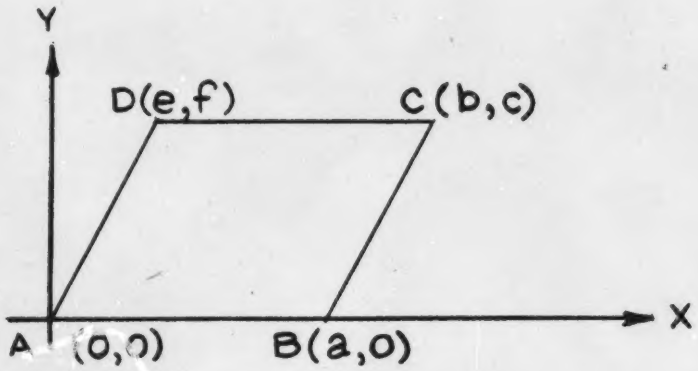
$AD = \sqrt{(c - 0)^2 + (b - 0)^2} = \sqrt{c^2 + b^2}$

$BC = \sqrt{(a + c - a)^2 + (b - 0)^2} = \sqrt{c^2 + b^2}$

$\therefore \overline{AD} \cong \overline{BC}$

Example 2:

Prove: If two sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.



$\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ By choice of coordinates for A and B,

slope of $\overleftrightarrow{AB} = 0$. Therefore, slope of $\overline{DC} = 0$

$$\frac{c - f}{b - e} = 0$$

$$c = f$$

$\overline{AB} \cong \overline{DC}$ $\quad AB = a$

$\therefore DC = a$

$b - e = a$

$b = a + e$

Coordinates of C: $(a + e, f)$

Therefore slope of $\overline{AD} = \frac{f - 0}{e - 0} = \frac{f}{e}$

slope of $\overline{BC} = \frac{f - 0}{a + e - a} = \frac{f}{e}$

$\therefore \overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$

V. Solution of Quadratic Equations

In their work with geometric figures, the students have had experience with problems whose solutions involve irrational numbers. It seems reasonable then, to introduce techniques such as the quadratic formula which make it possible to solve quadratic equations whose roots are not irrational.

Teachers may wish to begin by a reconsideration of simple quadratic equations such as $x^2 = 2$, and the nature of their solutions. In some cases, this discussion can lead to a proof that there is no rational number whose square is 2. This is a good example of the method of indirect proof, which some students have met in section I.

A. If the roots of a quadratic equation are rational, the equation or an equivalent equation can be solved by factoring over the set of integers and this method of solution should be reviewed. Students should recognize this as an application of the statement $ab = 0 \leftrightarrow a = 0$ or $b = 0$. The student should recognize that the zeros of the polynomial function defined by $y = ax^2 + bx + c$ are the roots of the polynomial equation $ax^2 + bx + c = 0$. These roots are represented graphically by the intersection of the parabola whose equation is $y = ax^2 + bx + c$ with the x-axis (the line $y = 0$). The parabola will intersect the x-axis when the roots are real. Students should be able to find the equation of the axis of symmetry and the coordinates of the turning point of the parabola.

B. If the polynomial $ax^2 + bx + c$ (or some multiple of it) is not factorable over the set of integers, then the roots of $ax^2 + bx + c = 0$ are not rational and other techniques are needed to find them. One such technique is the method of *completing the square*. This method should be introduced first with simple equations and then extended to more difficult ones.

Example 1:

Solve the quadratic equation $x^2 - 6x - 1 = 0$

$$x^2 - 6x - 1 = 0$$

$$x^2 - 6x = 1$$

$$x^2 - 6x + 9 = 1 + 9$$

$$(x - 3)^2 = 10$$

$$x - 3 = \pm\sqrt{10}$$

$$x = 3 \pm \sqrt{10}$$

(Note: In this section, solution set notation has been omitted. This is not meant to indicate that it should not be used.)

The perfect square trinomial employed for a particular equation is not unique and students should be free to choose any one which seems to make the work easier.

Example 2:

Solve $2x^2 - 5x - 1 = 0$

Choice 1:

$$2x^2 - 5x - 1 = 0$$

$$2x^2 - 5x = 1$$

$$x^2 - \frac{5x}{2} = \frac{1}{2}$$

$$x^2 - \frac{5x}{2} + \frac{25}{16} = \frac{1}{2} + \frac{25}{16}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{33}{16}$$

$$x - \frac{5}{4} = \frac{\pm\sqrt{33}}{4}$$

$$x = \frac{5 \pm \sqrt{33}}{4}$$

Choice 2:

$$2x^2 - 5x = 1$$

$$16x^2 - 40x = 8$$

$$16x^2 - 40x + 25 = 8 + 25$$

$$(4x - 5)^2 = 33$$

$$4x - 5 = \pm\sqrt{33}$$

$$4x = 5 \pm \sqrt{33}$$

$$x = \frac{5 \pm \sqrt{33}}{4}$$

In the second choice given above, fractions do not appear in the solution until the last step and for some students, this may simplify the work considerably.

C. The repetition of the same steps in the solution of each problem should suggest to the student that these steps applied to the general equation $ax^2 + bx + c = 0$ will result in a generalized solution, or formula. It is intended that only equations with real roots be introduced, although students' questions may necessitate some discussion of the case where $b^2 - 4ac < 0$.

Derivation 1: $ax^2 + bx + c = 0 \qquad (a \neq 0)$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Derivation 2: $ax^2 + bx + c = 0 \qquad (a \neq 0)$

$$ax^2 + bx = -c$$

$$4a^2x^2 + 4abx = -4ac$$

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

$$(2ax + b)^2 = b^2 - 4ac$$

$$2ax + b = \pm\sqrt{b^2 - 4ac}$$

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

D. The simultaneous solution of a quadratic-linear pair of equations can be approached from either the graphical or the algebraic point of view.

Example:

Solve:

$y = 2x^2 + 6x + 5$

$y = x + 3$

Solution:

$x + 3 = 2x^2 + 6x + 5$

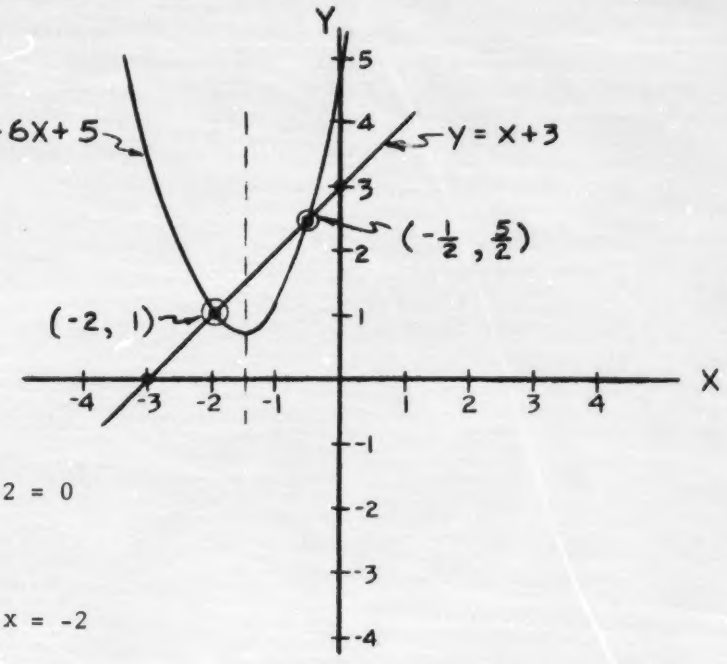
$0 = 2x^2 + 5x + 2$

$0 = (2x + 1)(x + 2)$

$2x + 1 = 0 \qquad x + 2 = 0$

$2x = -1$

$x = -\frac{1}{2} \qquad x = -2$



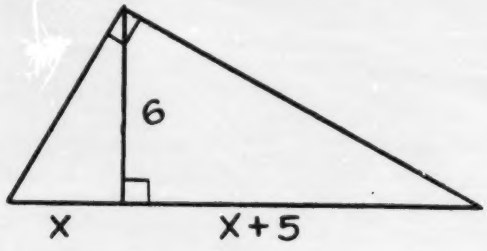
Substituting these values of x in the original linear equation, we obtain the solutions $(-\frac{1}{2}, \frac{5}{2})$ and $(-2, 1)$. These would then be checked in the quadratic equation.

If the quadratic equation obtained after substitution is not factorable, then the quadratic formula may be used. For graphical solutions, teachers may wish to limit the quadratic to a parabola.

E. Many of the concepts developed in the section on Euclidean geometry lead to problems whose solution involves quadratic equations. When selecting problems for class discussion the teacher should avoid choosing only those problems whose solutions involve rational numbers.

Example 1:

The altitude to the hypotenuse of a right triangle separates the hypotenuse into two parts, one of which measures 5 more than the other. If the measure of the altitude is 6, find the measure of the hypotenuse.



Let x = the length of the shorter segment of the hypotenuse.

$$\frac{x}{6} = \frac{6}{x + 5}$$

Using the theorem that the length of the altitude to the hypotenuse of a right triangle is the mean proportional between the lengths of the segments of the hypotenuse.

$$x^2 + 5x = 36$$

$$x^2 + 5x - 36 = 0$$

$$x = \frac{-5 \pm \sqrt{25 + 144}}{2}$$

$$x = \frac{-5 \pm 13}{2}$$

$$x = -9 \text{ or } x = 4$$

$$x = -9 \text{ rejected}$$

$\therefore$ The segments of the hypotenuse measure 4 and 9 and the measure of the hypotenuse is 13.

If in the problem given above, one segment had been 4 more than the other, the problem would have an irrational solution:

$$\frac{x}{6} = \frac{6}{x+4}$$

$$x^2 + 4x = 36$$

$$x^2 + 4x - 36 = 0$$

$$x = \frac{-4 \pm \sqrt{16 + 144}}{2}$$

$$x = \frac{-4 \pm \sqrt{160}}{2}$$

$$x = -2 \pm \sqrt{40} \quad \text{or}$$

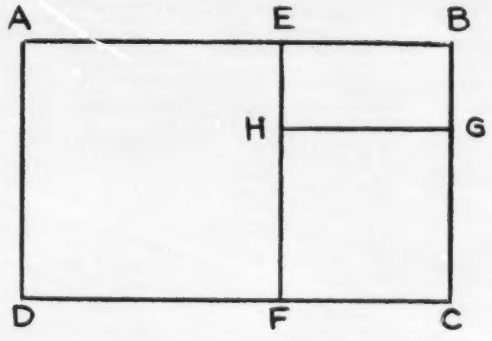
$$x = -2 \pm 2\sqrt{10}$$

$x = -2 - 2\sqrt{10}$ is rejected

$\therefore$ The segments of the hypotenuse have measure $-2 + \sqrt{40}$ and $2 + \sqrt{40}$ and the hypotenuse has measure $2\sqrt{40}$ (or $4\sqrt{10}$).

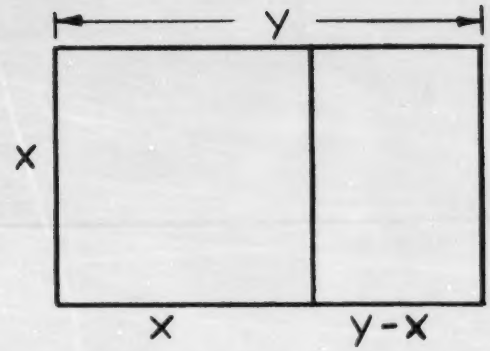
Example 2:

The golden rectangle, prized by the ancient Greeks as an especially pleasing shape, has the following property: If a square is cut off the rectangle, the figure which remains is a rectangle of the same proportions as the original.



After square AEFD has been cut off rectangle ABCD, the resulting figure, rectangle EBCF, is similar to the original rectangle ABCD. Likewise, rectangle EBGH is similar to rectangle EBCF (square HGCF has been removed) etc.

The ratio of the dimensions of such a rectangle, often called the golden ratio or golden section, appears frequently in art and in nature, and can be determined by use of the quadratic formula:



$$\frac{y}{x} = \frac{x}{y - x}$$

$$x^2 = y^2 - xy$$

$$x^2 + xy - y^2 = 0$$

The equation can be solved using the quadratic formula if we consider x to be the variable; y, since it is the coefficient of x to be b; and $-y^2$, since it is the term which does not contain x, to be c.

$$x = \frac{-y \pm \sqrt{y^2 + 4y^2}}{2}$$

$$x = \frac{-y \pm y\sqrt{5}}{2}$$

$$\frac{x}{y} = \frac{-1 + \sqrt{5}}{2} = \frac{\sqrt{5}}{2} - \frac{1}{2}$$ (rejecting the negative root and dividing by y)

Thus, the ratio of x to y, of the shorter dimension to the longer, is $\frac{\sqrt{5}}{2} - \frac{1}{2}$ (i.e., $x:y = \frac{\sqrt{5}}{2} - \frac{1}{2} : 1$) or about .6.

For many students it may be easier to determine the golden ratio by first considering a rectangle of length 1.

$$\frac{1}{x} = \frac{x}{1 - x}$$

$$x^2 = 1 - x$$

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 4}}{2}$$

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

$$\text{ratio of } \frac{\text{width}}{\text{length}} = \frac{x}{1} = \frac{-1 \pm \sqrt{5}}{2} = \frac{\sqrt{5}}{2} - \frac{1}{2} \approx 0.6$$

Since all golden rectangles are similar, the ratio of their dimensions must be constant.

VI. Probability, Permutations, and Combinations

A. This section represents a continuation of the material on probability and permutations begun in Course I of the program. Students will review the aspects of combinatorics already encountered (generally, students will not have done any work with combinations) and proceed to a new and deeper understanding of this area of mathematics. Teachers are referred to the outline given for Course I for further discussion of the reasoning involved in certain basic problems of probability theory, and for some suggestions on how to present this material.

As indicated, students should begin with a brief review of the sorts of considerations studied the previous year, as embodied, for example, in problems like the following:

(1) How many different "words" (i.e., permutations) can be made from the letters in PARIS?
(2) How many different words of exactly three letters can be made from the letters in PARIS?

This review can be extended to include problems like:

(3) How many different words of at *most* (or at least) three letters can be made from the letters in PARIS? (Note that this problem involves calculating the answer to three or four simpler problems and adding the results.)

At this time, the notation ${}_nP_n$ should be reviewed as well as the important idea that ${}_nP_n = n!$ and that

$${}_nP_r = n(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r+1) \qquad r < n$$

A series of examples would appear to be quite efficient for this purpose:

Example 1:

${}_7P_3 = 7 \times 6 \times 5 = 210$. In this case $n = 7$, $r = 3$, and $(n - r + 1) = 7 - 3 + 1 = 5$.

Example 2:

${}_{18}P_{11} = 18 \times 17 \times 16 \times \ldots \times 9 \times 8. = 1{,}270{,}312{,}245{,}200$

With additional examples of this type, and some time for individual investigation, most students should be prepared to accept the equivalence of

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{with} \quad {}_nP_r = n \cdot (n-1) \cdot (n-2) \cdot\ldots\cdot (n-r+1)$$

Example 3:

$${}_{18}P_{11} = \frac{18!}{(18-11)!} = \frac{18!}{7!} = \frac{18\cdot17\cdot16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

B. Until now, the permutation problems considered involved only permutations of things *without repetition*, that is, of things which were all different, for example, the letters in PARIS. As a bridge to our next topic, let us consider the problem of finding all possible permutations of the letters in EXETER. (Perhaps this is more a channel-crossing than a bridge, but in any case, the letter E appears three times.) This is a permutation problem in which the elements to be permuted are not all different.

Teachers may wish to discuss the solution to this problem in a variety of ways but one approach might be the following. Suppose that the E's in EXETER were differentiable from one another in some manner, say be numbering, in this fashion: $E_1X\ E_2T\ E_3R$, and suppose we tried to solve our problem by listing.

We would have, to start:

$E_1X\ E_2T\ E_3R$ $\quad$ $E_1X\ E_3T\ E_2R$ $\quad$ $E_3X\ E_1T\ E_2R$

$E_2X\ E_1T\ E_3R$ $\quad$ $E_2X\ E_3T\ E_1R$ $\quad$ $E_3X\ E_2T\ E_1R$

and

$E_1E_2E_3X\ T\ R$ $\quad$ $E_1E_3E_2X\ T\ R$ $\quad$ $E_3E_1E_2X\ T\ R$

$E_2E_1E_3X\ T\ R$ $\quad$ $E_2E_3E_1X\ T\ R$ $\quad$ $E_3E_2E_1X\ T\ R$

and so forth. But the E's in the original problem are *not* numbered, hence the first six "trial permutations" given are actually the same "final permutation." So are the next six. Indeed, every permutation in the final answer to our question would be one of six different "trial permutations" with the "numbers in." This means that the final answer is equal to the "trial answer" (total number of permutations pretending the E's were different) divided by six. Six, because, by previous work, the number of ways of permuting the three E's in whatever places they hold in the "word," is $3! = 3 \times 2 \times 1 = 6$. This and similar problems should help convince students that the number of permutations of n things, taken all at a time, with r of the things identical, is given by $\frac{n!}{r!}$.

Example 1:

The number of different words which can be made from QBBVRXT is $\frac{7!}{2!} = 2520$.

Example 2:

The number of different permutations of the symbols &&&&& is $\frac{5!}{5!} = 1$.

Example 3:

How many different ordered triples can be made using the components 0, 0, 1?

Solution: $\frac{3!}{2!} = 3$. The triples are (0, 0, 1), (0, 1, 0), (1, 0, 0).

The problem of finding the number of permutations of n things taken all at a time with r identical things of one kind, s identical things of another kind, t identical things of a third kind, and so forth, is more challenging, and ought perhaps, at this stage to be reserved for more skillful students. Here is a sample problem, followed by one possible approach to discussing its solution.

Problem: How many different words can be made from the letters in LONDON?

Solution: There are two O's and two N's. Suppose that the O's and N's are numbered, thus: L O_1N_1D O_2N_2. Then, for every permutation of the numbered O's, there are two for the numbered N's.

L O_1N_1D O_2N_2 L O_2N_1D O_1N_2

L O_1N_2D O_2N_1 L O_2N_2D O_1N_1

To reach a final solution, we must therefore divide the total number of trial permutations with the numbered letters by the total number of permutations just of O's and N's, which by previous work, including the Counting Principle, is 2! x 2! = 4. Our final answer is $\frac{6!}{2!2!} = 180$.

Example:

Find the total number, N, of words which can be made from the letters in MISSISSIPPI. Solution: There are four I's, four S's, and two P's.

Hence $N = \frac{11!}{4!4!2!} = 34{,}650$.

C. "Word" problems of the sort presented here are used only because they *do* involve the basic idea of *permutation* (rather than, say, combination) and are thus useful pedagogical tools. Teachers are encouraged to use, besides these, somewhat more realistic settings for this study; for example, those involving seating arrangements, class assignments in different subjects, electing students to different class offices, problems with dice or playing-cards, and so on.

A permutation is a selection where order, or arrangement, is significant. For example, #¢& and #&¢ are different permutations of those

three symbols. A *combination* is a selection where order does not matter: only the identity of the elements in the choice, or set, is significant. #¢& and #&¢ are thus different representations of the *same* combination.

Example:

> Indicate, by listing, all the combinations of three letters which can be made from the letters ABCD. Solution: ABC, ABD, ACD, BCD. In this solution, the combinations happened to be given alphabetically.

It is often helpful to think of combinations as *committees*. That is, in most circumstances, the order in which committee members are selected is of no significance; only their identity is. Jones, Mason and Kelly form the same committee no matter who was selected first, second, or third.

Three notations are frequently used for "the number of combinations of n things taken r at a time." They are:

$${}_nC_r \qquad \binom{n}{r} \qquad C(n, r)$$

As a start, students may be asked to consider the value of ${}_nC_n$, i.e., the number of combinations of n things taken all at a time. It should be clear that ${}_nC_n = 1$. This is, if you have n people from whom you must select a committee, and all n people must be members of the committee, then your options are clearly limited: you have only one way to make the choice.

Now, how are we to calculate ${}_nC_r$, for $r < n$? It is suggested that at first, examples with small numbers be chosen to give students ample opportunity to investigate the question on their own, by trial and error. Later, for the general case, a development similar to the following might be considered.

Problem: How many different committees of three can be chosen from among Sally, Mary, Tracey, and Binky (i.e., find ${}_4C_3$)? In this case, since the numbers are manageable, we could solve the problem by listing (same procedure as used to form different words from a set of letters), but let us try to arrive at a more general method. Call the potential committee members S, M, T, and B. Now we know there are ${}_4P_3 = 24$ different *permutations* of these symbols. They are:

SMT	SMB	STB	MTB
STM	SBM	SBT	MBT
MST	MSB	TSB	TMB
MTS	MBS	TBS	TBM
TSM	BMS	BTS	BMT
TMS	BSM	BST	BTM

Using an argument similar to the one given above for permutations with some objects identical, we can see that the number of distinct committees, or combinations, which can be formed is equal to the total number of *permutations* reduced in some way to account for all the listings which represent the same selection. But for each committee of three, say STB, there are 3! = 6 different permutations. Thus, to arrive at the number N of three-person committees (combinations) possible with four persons from

which to choose, it is necessary to divide the total number of permutations of four things taken three at a time by 3!, or $N = \frac{{}_4P_3}{3!}$

or $N = \frac{4!}{1!} \div 3!$ or $N = \frac{4!}{1!\ 3!}$

A similar argument for ${}_4C_2$ yields $N = \frac{{}_4P_2}{{}_2P_2} = \frac{4 \cdot 3}{2 \cdot 1}$ and for

${}_5C_3$ yields $N = \frac{{}_5P_3}{{}_3P_3} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1}$ and in general

$$ {}_nC_r = \frac{{}_nP_r}{{}_rP_r} $$

$$ {}_nC_r = \frac{n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - r + 1)}{r!} $$

$$ {}_nC_r = \frac{n!}{r!(n - r)!} $$

Example:

How many different committees can be formed from 10 people if Jones, one of the 10, *must* be on the committee? Solution: Jones is pre-selected, so the problem reduces, in effect, to filling the other nine places; that is, to computing $\binom{9}{3} = 84$.

It is evident that every choice of a combination of r things from a total of n things leaves a combination of n - r things unchosen. Hence it can be seen that ${}_nC_r = {}_nC_{n-r}$. Teachers may wish to introduce this theoretical idea before students actually compute ${}_nC_r$ for various n, r, as outlined here, but in any case, students should be given plenty of time to make this discovery, either by analysis of the situation, or by trial and error, on their own.

D. Problems involving permutations and combinations often occur in situations involving probability. It should be noted that some students experience difficulty in distinguishing situations in which probabilities should be added from those in which they should be multiplied and teachers may need to give special attention to this difficulty.

Problem 1: If two cards are drawn at random from a standard deck of 52, what is the probability of selecting:

1A: the ace and king of spades, in that order?
1B: any two aces?
1C: an ace and a king, in that order?
1D: an ace and a king, in either order?

Solutions using methods of this section:

For 1A: There are ${}_{52}P_2$ different selections possible of which one is a success. Answer: $\frac{1}{{}_{52}P_2}$.

For 1B: Using combinations, we may reason that there are ${}_{52}C_2$ different selections of two cards, and ${}_4C_2$ ways of choosing two aces from four. Answer: $\frac{{}_4C_2}{{}_{52}C_2}$. It is also possible to think of each outcome as a permutation and arrive, of course, at the same answer.

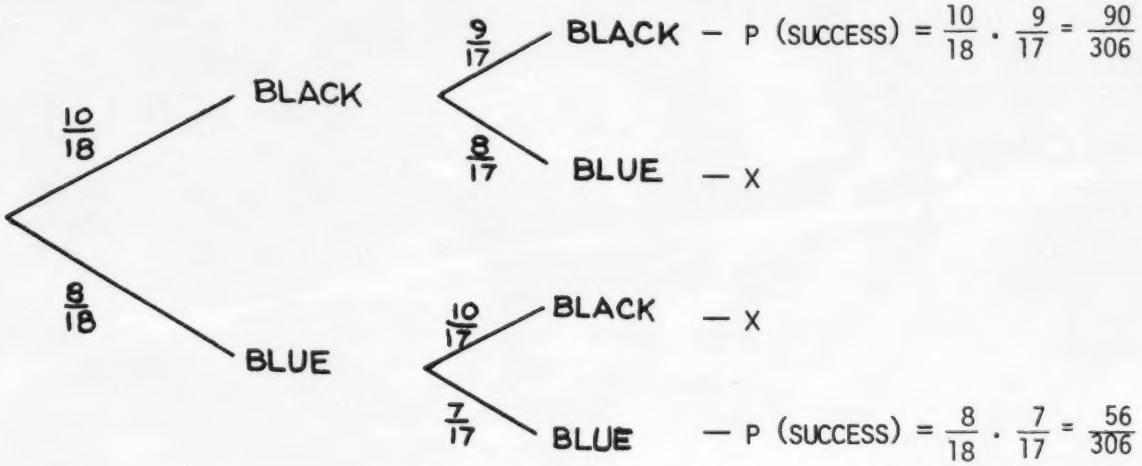
For 1C: There are ${}_4C_1$ ways of choosing the ace and ${}_4C_1$ ways of choosing the king. Answer: $\frac{{}_4C_1 \cdot {}_4C_1}{{}_{52}C_2}$

For 1D: The answer ought to be double that in 1C since we are adding to the list of possible successes all the king-ace combinations.

Many of the problems encountered in Course I which could be dealt with using the "tree" approach can also be dealt with, perhaps more efficiently, using the methods of this section.

Problem 2: (from Course I): Mr. Smythe reaches into his sock-drawer one dim December morning and chooses two socks, not examining them as carefully as he might. If the drawer contained ten black socks and eight navy blue ones, what is the probability Mr. Smythe pulled out a pair?

Solution 1: (from Course I).



$$P\ (\text{SUCCESS}) = P\ (2\ \text{BLACK}) + P\ (2\ \text{BLUE}) = \frac{90}{306} + \frac{56}{306} = \frac{146}{306} = \frac{73}{153}$$

Solution 2: There are $C(18, 2)$ ways of making the sock-selection, $C(10, 2)$ ways of selecting a pair of black socks and $C(8, 2)$ ways of selecting a pair of blue socks. Hence the required probability is

$$\frac{C(10, 2)}{C(18, 2)} + \frac{C(8, 2)}{C(18, 2)} = \frac{C(10, 2) + C(8, 2)}{C(18, 2)} = \frac{45 + 28}{153} = \frac{73}{153}$$

Problem 3: There are four Republicans and five Democrats on a certain committee. If three committee members are chosen at random to form a subcommittee, what is the probability that all three will be Democrats? Solution: There are $\binom{9}{3}$ different ways of choosing three people from nine, and $\binom{5}{3}$ different ways the three Democrats can be selected.

Answer: $$\frac{\binom{5}{3}}{\binom{9}{3}} = \frac{5}{42}$$

Problem 4: How many different hands are there at bridge? This can be considered a combination problem since a bridge hand does not depend on the order in which the cards are dealt. Answer: C(52, 13).

It will be noted that reaching a "final answer" to certain combinatoric problems can be relatively tedious and time-consuming, consisting as it does chiefly of arithmetic. Teachers may wish to have students complete several problems down to the final answer, to have them get an idea of the size of the numbers in question, or for practice in computation, but it is suggested that this aspect of the work be deemphasized except in cases where teachers feel their students will derive a particular benefit from the calculations. Some teachers may wish to use the electronic calculator as an aid in some of the calculations of this section.

Needless to say, teachers who wish to pursue the topic of combinations further will find plenty of material, and additional exercises, in the many books available on the subject of probability. Some teachers may wish to deal with the Binomial theorem (see Section VII) as one example of the use of the mathematical materials developed in this section.

VII. Possible Optional Topics

As time permits, teachers may wish to explore the following topics.

A. Binomial theorem
B. Space visualization
C. Number theory
D. Introduction to trigonometry
E. Matrices
F. Computers